AN ANALYTIC FORMULA FOR THE EQUILIBRIUM TEMPERATURE

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An exact formula is obtained for the equilibrium temperature of a surface in a stream of hot gas. The heat-transfer coefficient is supposed known.

In the study of aerohydrodynamic heating it is frequently necessary to know the equilibrium temperature of a surface, T_{eq}, determined by the heat balance between convective heat flow and radiative transfer:

$$\alpha (T_e - T_{eq}) = \varepsilon\sigma T_{eq}^4 . \tag{1}$$

Up to the present time this equation has been solved graphically, by the method of successive approximations, or by approximate formulas [1]. We present the derivation of an exact analytic formula for the equilibrium temperature of a surface when the heat-transfer coefficient α is known.

We transform Eq. (1) to the form

$$T_{eq}^4 + \frac{\alpha T_{eq}}{\varepsilon\sigma} - \frac{\alpha T_e}{\varepsilon\sigma} = 0. \tag{1'}$$

The roots of this equation coincide with the roots of two quadratic equations [2]

$$T_{eq}^2 + A\frac{T_{eq}}{2} + y - \frac{\frac{\alpha}{\varepsilon\sigma}}{A} = 0, \tag{2}$$

where $A = \pm\sqrt{8y}$ and y is the real root of the cubic equation

$$8y^3 + 8\frac{\alpha T_e}{\varepsilon\sigma}y - \left(\frac{\alpha}{\varepsilon\sigma}\right)^2 = 0, \tag{3}$$

which reduces to

$$y^3 + 3py - 2q = 0. \tag{3'}$$

Since the discriminant $D = q^2 + p^3 > 0$, Equation (3') has one real root which can be found from the Cardan solution: $y = u + v$, where

$$u = \sqrt[3]{-q + \sqrt{q^2 + p^3}}, \quad v = \sqrt[3]{-q - \sqrt{q^2 + p^3}}.$$

Consequently

$$y = \left[\frac{\alpha^2}{16\,\varepsilon^2\sigma^2} + \sqrt{\left(\frac{\alpha^2}{16\,\varepsilon^2\sigma^2}\right)^2 + \left(\frac{\alpha T_e}{3\,\varepsilon\sigma}\right)^3}\right]^{1/3} + \left[\frac{\alpha^2}{16\,\varepsilon^2\sigma^2} - \sqrt{\left(\frac{\alpha^2}{16\,\varepsilon^2\sigma^2}\right)^2 + \left(\frac{\alpha T_e}{3\,\varepsilon\sigma}\right)^3}\right]^{1/3}. \tag{4}$$

From (2) the equilibrium temperature is

$$T_{eq} = -\frac{A}{4} \pm \sqrt{\frac{A^2}{16} - \left(y - \frac{\alpha}{A\varepsilon\sigma}\right)}. \tag{5}$$

For $A = -\sqrt{8y}$ we obtain the two complex conjugate roots

$$T_{eq} = \sqrt{\frac{y}{2}} \pm \sqrt{\frac{y}{2} - \left(y + \frac{\alpha}{\varepsilon\sigma\sqrt{8y}}\right)} = \sqrt{\frac{y}{2}} \pm \sqrt{-\frac{y}{2} - \frac{\alpha}{\varepsilon\sigma\sqrt{8y}}}. \tag{6}$$

These roots do not correspond to the required real physical solution.

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For positive $A = \sqrt{8y}$

$$T_{eq} = -\sqrt{\frac{y}{2}} \pm \sqrt{-\frac{y}{2} + \frac{\alpha}{\varepsilon\sigma\sqrt{8y}}}\,. \tag{7}$$

Since a positive solution is required and y, α, ε, and σ are positive at the equilibrium temperature, there remains the single real positive root

$$T_{eq} = -\sqrt{\frac{y}{2}} + \sqrt{-\frac{y}{2} + \frac{\alpha}{\varepsilon\sigma\sqrt{8y}}}\,. \tag{8}$$

Similarly an analytic formula can be obtained for the steady-state temperature of a wall heated by a stream of hot gas on the outside, and cooled from the inside. Thus, for example, for the steady removal of heat through a wall with a heat-transfer coefficient α_{in}, the heat-balance equation has the form:

$$\alpha(T_e - T_{out}) = \varepsilon\sigma T_{out}^4 + \alpha_{in}(T_{out} - T_{in}), \tag{9}$$

and

$$\alpha_{in} = \frac{\lambda}{\delta}, \tag{10}$$

$$T_{out} = -\sqrt{\frac{y_1}{2}} + \sqrt{-\frac{y_1}{2} + \frac{\alpha + \alpha_{in}}{\varepsilon\sigma\sqrt{8y_1}}}\,. \tag{11}$$

where

$$y_1 = \left[\frac{(\alpha+\alpha_{in})^2}{16\,\varepsilon^2\sigma^2} + \sqrt{\left[\frac{(\alpha+\alpha_{in})^2}{16\,\varepsilon^2\sigma^2}\right]^2 + \left(\frac{\alpha T_e + \alpha_{in} T_{in}}{3\,\varepsilon\sigma}\right)^3}\,\right]^{1/3} + \left[\frac{(\alpha+\alpha_{in})^2}{16\,\varepsilon^2\sigma^2} - \sqrt{\left[\frac{(\alpha+\alpha_{in})^2}{16\,\varepsilon^2\sigma^2}\right]^2 + \left(\frac{\alpha T_e + \alpha_{in} T_{in}}{3\,\varepsilon\sigma}\right)^3}\,\right]^{1/3}. \tag{12}$$

NOTATION

- T_{in} is the temperature of the inside surface;
- T_e is the recovery temperature;
- T_{out} is the temperature of the outside surface;
- T_{eq} is the equilibrium temperature;
- α is the coefficient for heat transfer from the hot gas to the surface;
- α_{in} is the coefficient for heat transfer through the wall;
- δ is the wall thickness;
- ε is the emissivity;
- λ is the thermal conductivity;
- σ is the Stefan–Boltzmann constant.

LITERATURE CITED

1 N. F. Krasnov, Aerodynamics of Bodies of Revolution, Elsevier, New York (1970).

2. V. I. Smirnov, A Course in Higher Mathematics [in Russian], Vol. 1, GIFML, Moscow (1961).